One-dimensional polymers in random environments: the ballistic regime

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Outline

1. The model
2. Target problem
3. Phase diagram
4. Newly added results for the ballistic regime
5. Future works

The model

Let $(S_n)_{n\geqslant 0}$ be the d-dimensional simple random walk, $(\omega_x)_{x\in\mathbb{Z}^d}$ be iid random fields with mean 0 and finite exp moments. The d-dim Wiener sausages in disordered systems with $R_N := |\mathcal{R}_N|$, $\mathcal{R}_N := \{S[1,N]\}$

$$H_N := \sum_{x\in\mathbb{Z}^d} (\beta\omega_x - h)\, 1_{x\in\mathcal{R}_N}$$

$\beta \geqslant 0$, $h > 0$.

$$\frac{d\mathrm{P}^{\omega}_{N,\beta,h}}{d\mathrm{P}}(S) := \frac{1}{Z^{\omega}_{N,\beta,h}} \exp\left(\sum_{x\in\mathbb{Z}^d} (\beta\omega_x - h) 1_{x\in\mathcal{R}_N}\right),$$

$$Z^{\omega}_{N,\beta,h} := \mathrm{E}\left[\exp\left(\sum_{x\in\mathbb{Z}^d} (\beta\omega_x - h) 1_{x\in\mathcal{R}_N}\right)\right] = \mathrm{E}\left[\exp\left(\beta \sum_{x\in\mathcal{R}_N} \omega_x - h|\mathcal{R}_N|\right)\right]$$

Endpoint-fluctuation exponent

We now wish to understand the typical behavior of polymer trajectories $(S_0, \dots, S_N)$ under the polymer measure $\mathrm{P}^{\omega}_{N,\beta,h}$. Two important quantities that we are interested in are

- the *endpoint* exponent ξ, loosely defined as $\mathbb{E}\mathrm{E}^{\omega}_{N,\beta,h}|S_N| \approx N^{\xi}$;
- the *fluctuation* exponent χ, loosely defined as $|\log Z_{N,\beta,h} - \mathbb{E}[\log Z_{N,\beta,h}]| \approx N^{\chi}$.

$$Z^{\omega}_{N,\beta,h} := \mathrm{E}\left[\exp\left(\sum_{x\in\mathbb{Z}^d}(\beta\omega_x - h)1_{x\in\mathcal{R}_N}\right)\right] = \mathrm{E}\left[\exp\left(\beta\sum_{x\in\mathcal{R}_N}\omega_x - h|\mathcal{R}_N|\right)\right]$$

If $\beta = 0$ and $h > 0$, then we recover a random walk penalized by its range. This model is by now quite well understood: the random walk folds itself in a ball of radius $N^{1/(d+2)}$ ($\xi = \frac{1}{d+2}$), see DV79,Bolt94,BC18,DFSX18 (these works mostly focus on the case of dimension $d \geqslant 2$).

$$Z^{\omega}_{N,\beta,h} := \mathrm{E}\Bigg[\exp\Bigg(\sum_{x\in\mathbb{Z}^d}(\beta\omega_x - h)1_{x\in\mathcal{R}_N}\Bigg)\Bigg] = \mathrm{E}\Bigg[\exp\Bigg(\beta\sum_{x\in\mathcal{R}_N}\omega_x - h|\mathcal{R}_N|\Bigg)\Bigg]$$

If $\beta > 0$ and $h = 0$, then we obtain a model for the range in the random environment from Huang19. We will see that it is indeed the case in dimension $d = 1$, where we find that the random walk stretches up to a distance $N^{2/3}$ ($\xi = \frac{2}{3}$) as predicted in Huang19.

Transition: folding and stretching

$$H_N := \sum_{x \in \mathbb{Z}^d} (\beta \omega_x - h) \, 1_{x \in \mathcal{R}_N}$$

folded phase $(h > 0, \beta = 0)$ v.s. unfolded phase $(h = 0, \beta > 0)$

Consider parameters β and h that depend on the size of the system, *i.e.* $\beta := \beta_N$ and $h := h_N$.

There are then some sophisticated balances between the energy gain, the range penalty and the entropy cost as we tune β_N and h_N.

We set

$$\beta_N := \hat{\beta} \, N^{-\gamma} \,, \qquad \text{and} \qquad h_N := \hat{h} \, N^{-\zeta} \,, \tag{1}$$

where $\gamma, \zeta \in \mathbb{R}$ describe the asymptotic behavior of β_N, h_N, and $\hat{\beta}, \hat{h} > 0$ are two fixed parameters.

Assumption

Our main assumption on the environment is that ω_x is in the domain of attraction of some α-stable law, with $\alpha \in (0, 2]$, $\alpha \neq 1$. More precisely, we make the following assumption.
If $\alpha = 2$ we assume that $\mathbb{E}[\omega_0] = 0$ and $\mathbb{E}[\omega_0^2] = 1$. If $\alpha \in (0, 1) \cup (1, 2)$ we assume that $\mathbb{P}(\omega_0 > t) \sim p\, t^{-\alpha}$ and $\mathbb{P}(\omega_0 < -t) \sim q\, t^{-\alpha}$ as $t \to \infty$, with $p + q = 1$ (and $p > 0$). Moreover, if $\alpha \in (1, 2)$, we also assume that $\mathbb{E}[\omega_0] = 0$.
Let us stress that assumption ensures that:

- if $\alpha = 2$, then ω_i is in the normal domain of attraction, so that $(\frac{1}{\sqrt{n}} \sum_{i=un}^{vn} \omega_i)_{u \leqslant 0 \leqslant v}$ converges to a two-sided (standard) Brownian Motion.
- if $\alpha \in (0, 1) \cup (1, 2)$, then ω_i is in the domain of attraction of some non-Gaussian stable law and $(\frac{1}{n^{1/\alpha}} \sum_{i=un}^{vn} \omega_i)_{u \leqslant 0 \leqslant v}$ converges to a two-sided α-stable Lévy process.

Enery, range, entropy

$$\log \mathrm{P}(|\mathcal{R}_N| \approx N^{\xi}) \approx \log \mathrm{P}\Big(\max_{1\leqslant n\leqslant N} |S_n| \approx N^{\xi}\Big) \approx \begin{cases} -N^{2\xi-1}, & \text{if } \xi \geqslant \frac{1}{2}, \\ -N^{1-2\xi}, & \text{if } \xi \leqslant \frac{1}{2}. \end{cases} \tag{2}$$

If $\xi > 1/2$, this corresponds to a "stretching" of the random walk, whereas when $\xi < 1/2$, this corresponds to a "folding" of the random walk.
Then, if the endpoint fluctuations are of order N^{ξ} ($|\mathcal{R}_N| \approx N^{\xi}$), we get under Assumption, and in view of (1), that

$$\beta_N \sum_{x\in\mathcal{R}_N} \omega_x \approx N^{\frac{\xi}{\alpha}-\gamma}, \qquad h_N|\mathcal{R}_N| \approx N^{\xi-\zeta}. \tag{3}$$

If endpoint fluctuations are of order N^{ξ}, we have that

$$\log Z^{\omega}_{N,\beta_N,h_N} \approx N^{\frac{\xi}{\alpha}-\gamma} - N^{\xi-\zeta} - \begin{cases} N^{1-2\xi} & \text{if } \xi \leqslant 1/2\,, \\ N^{2\xi-1} & \text{if } \xi \geqslant 1/2\,. \end{cases} \tag{4}$$

There are three main possibilities:

(i) there is a "disorder"-"entropy" balance (and the "range" term is negligible);

(ii) there is a "range"-"entropy" balance (and the "energy" term is negligible);

(iii) there is a "range"-"disorder" balance (and the "entropy" term is negligible).

To summarize, all three regimes can occur (depending on γ, ζ) if $\alpha \in (1, 2]$; on the other hand, regime (iii) disappears if $\alpha \in (0, 1)$, and regime (i) disappears if $\alpha \in (0, \frac{1}{2})$. We now determine for which values of γ, ζ one can observe the different regimes above: we consider the three subcases $\alpha \in (1, 2]$, $\alpha \in (\frac{1}{2}, 1)$ and $\alpha \in (0, \frac{1}{2})$ separately.

Phase diagram for $\alpha \in (0, \frac{1}{2})$

$R_1 : \xi = \frac{1}{2}$
$R_3 : \xi = 1$
$R_5 : \xi = \frac{1+\zeta}{3}$
$R_6 : \xi = 0$

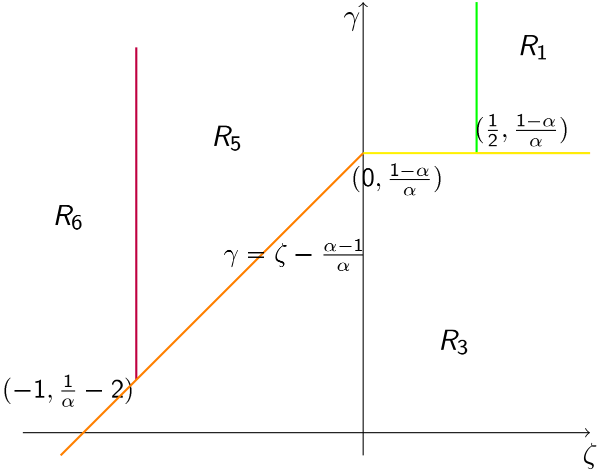


Figure: Phase diagram in the case $\alpha \in (0, 1/2)$. Compared to Figure 2, the region R_2 no longer exists.

Phase diagram for $\alpha \in (\frac{1}{2}, 1)$

$R_1 : \xi = \frac{1}{2}$
$R_2 : \xi = \frac{\alpha}{2\alpha - 1}(1 - \gamma)$
$R_3 : \xi = 1$
$R_5 : \xi = \frac{1+\zeta}{3}$
$R_6 : \xi = 0$

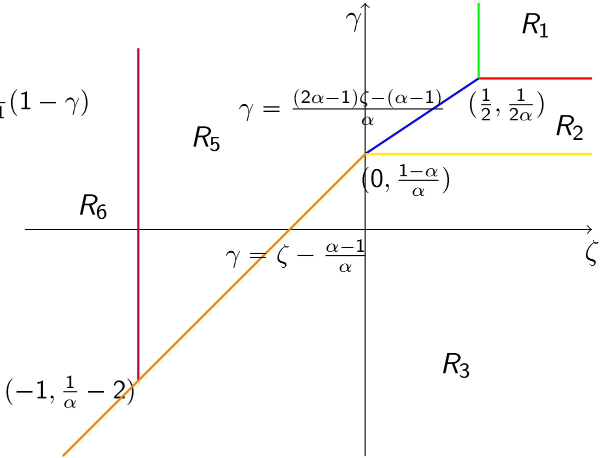
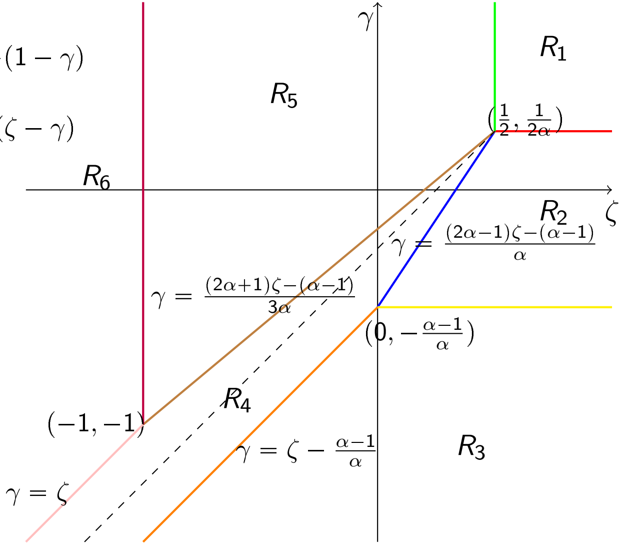


Figure: Phase diagram in the case $\alpha \in (1/2, 1)$. Compared to Figure **??**, the region R_4 no longer exists.

$R_1 : \xi = \frac{1}{2}$
$R_2 : \xi = \frac{\alpha}{2\alpha-1}(1-\gamma)$
$R_3 : \xi = 1$
$R_4 : \xi = \frac{\alpha}{\alpha-1}(\zeta-\gamma)$
$R_5 : \xi = \frac{1+\zeta}{3}$
$R_6 : \xi = 0$



Diffusive region

Theorem (**Region 1**)

Assume that (1) *holds with*

$$\begin{cases} \gamma > \frac{1}{2\alpha} \text{ and } \zeta > \frac{1}{2}, & \text{if } \alpha \in (\frac{1}{2}, 1) \cup (1, 2], \\ \gamma > \frac{1-\alpha}{\alpha} \text{ and } \zeta > \frac{1}{2}, & \text{if } \alpha \in (0, \frac{1}{2}). \end{cases}$$

Then, $(S_n)_{0 \leqslant n \leqslant N}$ *has endpoint fluctuations of order* $\sqrt{N}$ *under* $\mathrm{P}^{\omega}_{N,\beta_N,h_N}$ *(i.e.* $\xi = \frac{1}{2}$*), and we have the following convergence in probability*

$$Z^{\omega}_{N,\beta_N,h_N} \xrightarrow{\mathbb{P}} 1\,.$$

Theorem (**Region 2**)

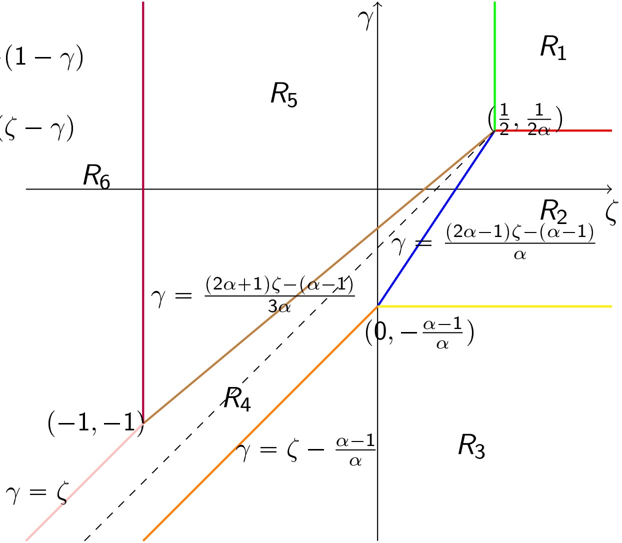
$(S_n)_{0\leqslant n\leqslant N}$ has endpoint fluctuations of order N^{ξ} with $\xi = \frac{\alpha}{2\alpha-1}(1-\gamma) \in (\frac{1}{2}, 1)$ under $\mathrm{P}^{\omega}_{N,\beta_N,h_N}$, and we have the following convergence in distribution

$$\frac{1}{N^{\frac{\xi}{\alpha}-\gamma}} \log Z^{\omega}_{N,\beta_N,h_N} \xrightarrow{d} \mathcal{W}_{R_2} := \sup_{u\leqslant 0\leqslant v} \left\{ \hat{\beta}(X_v - X_u) - I(u,v) \right\}, \qquad (5)$$

where $I(u,v) = \frac{1}{2}(|u| \wedge |v| + v - u)^2$ for $u \leqslant 0 \leqslant v$. Moreover, we have that $\mathcal{W}_{R_2} \in (0, +\infty)$, $\mathbb{P}$-almost surely.

Let us stress that the case $\alpha = 2$, $\beta = \beta_N \equiv \beta > 0$ and $h \equiv 0$ corresponds to the case $\gamma = 0$ and $\zeta = +\infty$: we find in that case that the endpoint fluctuation exponent is $\xi = \frac{2}{3}$.

$R_1 : \xi = \frac{1}{2}$
$R_2 : \xi = \frac{\alpha}{2\alpha-1}(1-\gamma)$
$R_3 : \xi = 1$
$R_4 : \xi = \frac{\alpha}{\alpha-1}(\zeta-\gamma)$
$R_5 : \xi = \frac{1+\zeta}{3}$
$R_6 : \xi = 0$



Theorem (**Region 4**)

$(S_n)_{0\leqslant n\leqslant N}$ has endpoint fluctuations of order N^{ξ} with $\xi = \frac{\alpha}{\alpha-1}(\zeta-\gamma) \in (0,1)$ under $\mathrm{P}^{\omega}_{N,\beta_N,h_N}$, and we have the following convergence in distribution

$$\frac{1}{N^{\xi-\zeta}} \log Z^{\omega}_{N,\beta_N,h_N} \xrightarrow{d} \mathcal{W}_{R_4} := \sup_{u\leqslant 0\leqslant v} \left\{ \hat{\beta}(X_v - X_u) - \hat{h}(v-u) \right\}. \quad (6)$$

Theorem (**Region 5**)

$(S_n)_{0\leqslant n\leqslant N}$ has endpoint fluctuations of order N^{ξ} with $\xi = \frac{1+\zeta}{3} \in (0, \frac{1}{2})$ under $\mathrm{P}^{\omega}_{N,\beta_N,h_N}$, and we have the following convergence in probability

$$\frac{1}{N^{\xi-\zeta}} \log Z^{\omega}_{N,\beta_N,h_N} \xrightarrow{\mathbb{P}} -\frac{3(4\hat{h}\pi)^{2/3}}{8} = \sup_{r\geqslant 0} \left\{ -\hat{h}r - \frac{\pi^2}{8r^2} \right\}. \quad (7)$$

Results in the case $\hat{h} < 0$

$$H_N := \sum_{x\in\mathbb{Z}^d} (\beta_N \omega_x - h_N)\, 1_{x\in\mathcal{R}_N},$$

$$\beta_N := \hat{\beta}\, N^{-\gamma} \quad \text{and} \quad h_N := \hat{h}\, N^{-\zeta}.$$

Energy and range both encourage outward, $\xi \geqslant 1/2$.
Heuristics:

$$\beta \int_0^{M_T} W(dx) + hM_T \approx \beta (M_T)^{1/2} + hM_T$$

The endpoint behavior is at a speed hT, and $T^{-1} \log Z_T \to \frac{1}{2} h^2$.
Moreover,

$$\frac{1}{\sqrt{T}} \left(\log Z_T - \frac{1}{2} h^2 T \right) \approxeq \beta \frac{W((0, hT])}{\sqrt{T}}.$$

$R_1 : \xi = \frac{1}{2}$
$R_2 : \xi = \frac{\alpha}{2\alpha-1}(1-\gamma)$
$R_3 : \xi = 1$
$\tilde{R}_4 : \xi = 1-\zeta$
$\tilde{R}_5 : \xi = 1$

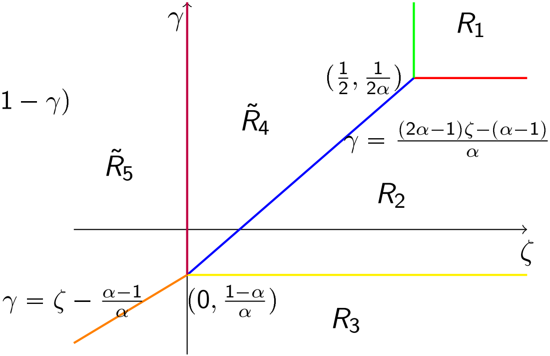


Figure: Phase diagram for $\hat{h} < 0$, in the case $\alpha \in (1/2, 2]$.

Theorem (**Region $\tilde{R}_4$: $\xi = 1 - \zeta \in (\frac{1}{2}, 1)$**)

$$\lim_{N \to +\infty} \frac{1}{N^{\xi - \zeta}} \log Z^{\omega}_{N,\beta_N,h_N} = \frac{1}{2}\hat{h}^2, \qquad \hat{\mathbb{P}}\text{-a.s.}$$

Additionally, let us consider, for $\varepsilon > 0$, the two events

$$\mathcal{B}_N^{\pm,\varepsilon} := \left\{ \sup_{t \in [0,1]} \left| N^{-\xi} S_{\lfloor tN \rfloor} \pm \hat{h}\, t \right| \leqslant \varepsilon \right\}$$

Then for any $\varepsilon > 0$, we have and $\hat{\mathbb{P}}$-a.s.

$$\lim_{\varepsilon \downarrow 0} \lim_{N \to +\infty} \mathrm{P}^{\hat{\omega}}_{N,\beta_N,h_N}\left(\mathcal{B}_N^{+,\varepsilon}\right) = \begin{cases} 1_{\{X^{(1)}_{|\hat{h}|} > X^{(2)}_{|\hat{h}|}\}} & \text{if } \gamma < \frac{1-\zeta}{\alpha}, \\ \dfrac{\exp\left(\hat{\beta} X^{(1)}_{|\hat{h}|}\right)}{\exp\left(\hat{\beta} X^{(1)}_{|\hat{h}|}\right) + \exp\left(\hat{\beta} X^{(1)}_{|\hat{h}|}\right)} & \text{if } \gamma = \frac{1-\zeta}{\alpha}, \\ \frac{1}{2} & \text{if } \gamma > \frac{1-\zeta}{\alpha}. \end{cases}$$

Discussion

Heuristics:

$$\beta \int_0^{M_T} W(dx) + hM_T$$

If $B_T \approx hT + \mathcal{Y}T^{2/3}$,

$$\log Z_T \approx \beta W_1(hT + \mathcal{Y}T^{2/3}) + h(hT + \mathcal{Y}T^{2/3}) - \frac{(hT + \mathcal{Y}T^{2/3})^2}{2T}$$

$$= \beta W_1(hT) + \beta W_2(\mathcal{Y}T^{2/3}) - \frac{1}{2}\mathcal{Y}^2 T^{1/3} + \frac{1}{2}h^2 T.$$

Thus,

$$\log Z_T - \frac{1}{2}h^2 T - \beta W_1(hT) \approx T^{1/3} \sup_{v \in \mathbb{R}} \left\{ W_2(v) - \frac{1}{2}v^2 \right\}$$

Technical estimates

Lemma

If $\xi \in (\frac{1}{2}, 1)$ *then for any* $u, v \geqslant 0$ *we have that*

$$\lim_{N\to\infty} -\frac{1}{N^{2\xi-1}} \log \mathrm{P}\Big(M_N^- \leqslant -uN^{\xi}; M_N^+ \geqslant vN^{\xi}\Big) = \frac{1}{2}(u \wedge v + u + v)^2 . \quad (8)$$

Lemma

For any $u, v \geqslant 0$*, we have that*

$$\lim_{N\to\infty} -\frac{1}{N} \log \mathrm{P}\big(M_N^- \leqslant -uN; M_N^+ \geqslant vN\big) = \kappa\big(u \wedge v + u + v\big) ,$$

where $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ *is the LDP rate function for the simple random walk, that is* $\kappa(t) := \frac{1}{2}(1+t)\log(1+t) + \frac{1}{2}(1-t)\log(1-t)$ *if* $0 \leqslant t \leqslant 1$ *and* $\kappa(t) = +\infty$ *if* $t > 1$.

Technical estimates

Lemma

Let $(X_v^{(1)})_{v\geqslant 0}$ *and* $(X_u^{(2)})_{u\geqslant 0}$ *be two independent* α*-stable Lévy processes and* $f:(\mathbb{R}_+)^2\to\mathbb{R}$ *be any function. Denote* $\mathcal{I}_{a,b}^{c,d}:=[c,d]\times[a,b]$*. Then for any two disjoint rectangles* $\mathcal{I}_{a,b}^{c,d}$ *and* $\mathcal{I}_{a',b'}^{c',d'}$*, we have that*

$$\mathbb{P}\left(\sup_{(u,v)\in\mathcal{I}_{a,b}^{c,d}}\left\{X_v^{(1)}+X_u^{(2)}+f(u,v)\right\}=\sup_{(u',v')\in\mathcal{I}_{a',b'}^{c',d'}}\left\{X_{v'}^{(1)}+X_{u'}^{(2)}+f(u',v')\right\}\right)$$
$$=0.$$

Uniqueness of the maximizer.

Technical estimates

Lemma

Let $\Sigma_\ell^* := \sup_{0\leqslant j\leqslant \ell} |\Sigma_j^-| + \sup_{0\leqslant j\leqslant \ell} |\Sigma_j^+|$ *. Then, for* $\alpha \in (0,1)\cup(1,2]$*, there exists a constant* $c \in (1,+\infty)$ *such that for any* $\mathsf{T} > 0$ *and any* ℓ *we have*

$$\mathbb{P}\left(\Sigma_\ell^* > \mathsf{T}\right) \leqslant c\,\ell\,\mathsf{T}^{-\alpha}\,. \tag{9}$$

Also, $\mathbb{P}$*-a.s. there is a constant* $C = C(\omega)$ *such that* $\Sigma_\ell^* \leqslant C\ell^{1/\alpha}(\log_2 \ell)^{2/\alpha}$ *for all* $\ell \geqslant 1$*.*

Etemadi's inequality.

Technical estimates

Lemma

Let $(\alpha_N(t))_{N\geqslant 1}$ *be a sequence of càd-làg paths on* $[0,\infty)$ *that converges to a càd-làg path* $\alpha(t)$ *for the Skorokhod distance* d_0 *(cf. [?]). Suppose that* α *is continuous at* u*. Then for any* $\epsilon, \delta > 0$*, there exists* $N_0 = N_0(u,\varepsilon,\delta) > 0$ *such that for all* $N \geqslant N_0$*,*

$$|\alpha_N(u) - \alpha(u)| < \varepsilon, \tag{10}$$

$$\sup_{v\in[u,u+\delta]} |\alpha_N(v) - \alpha_N(u)| < \varepsilon + \sup_{v\in[u,u+\delta+\varepsilon]} |\alpha(v) - \alpha(u)|. \tag{11}$$

Future works

- Finite temperature: localized/delocalized
- Strong disoder $d = 2$.
- Diffusive phase for $d \geqslant 3$.
- Fluctuations of $\log Z_N$

Discussion with directed random polymers

- $S = \{S_n\}_n$ is a random walk. That is, $S_0 = 0$ and $S_n = \sum_{j=1}^n X_j$. $\omega = \{\omega(i,j)\}$ is an i.i.d. environment under the law $\mathbb{P}$. $\beta \geqslant 0$.
- $$\frac{\mathrm{d}P_{N,\beta}(S)}{\mathrm{d}P} = \frac{1}{Z_{N,\beta}} \exp\left(\beta \sum_{j=1}^{N} \omega(j, S_j)\right)$$
- $Z_{N,\beta} = E(e^{\beta \sum_{j=1}^{N} \omega(j,S_j)})$.
- strong disorder for all $\beta > 0$ for $d = 1, 2$ and large β for $d \geqslant 3$.
- Random growth models: longest increasing subsequence, last passage percolation, simple exclusion process, max eiganvalue of GUE, etc..

$d = 2$

Continuous directed random polymers

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \beta u \cdot \dot{W}(t,x), \quad t \geqslant 0, x \in \mathbb{R}^2.$$

Continuous parabolic Anderson model

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \beta u \cdot \dot{W}(x), \quad t \geqslant 0, x \in \mathbb{R}^2.$$

Continuous disordered range model

$$\partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) + \beta u \diamond \dot{W}(x), \quad t \geqslant 0, x \in \mathbb{R}^2.$$